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A GEOMETRICAL METHOD FOR DETERMINING THE PARAMETERS OF MOTION OF SPACECRAFT IN A NEWTONIAN FIELD OF FORCE

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A GEOMETRICAL METHOD FOR DETERMINING THE PARAMETERS OF MOTION OF SPACECRAFT IN A NEWTONIAN FIELD OF FORCE

G. V. Mozhayev

A method is proposed for determining the parameters of motion using geometrical constructions based on the properties of a velocity hodograph. This method made it possible to derive analytical solutions which are difficult to find using classical integrals.

/1103*

Notations. r, V, θ, ϑ are the distance of the spacecraft from the center of gravitation, velocity, angle of departure and the true anomaly respectively; a, p, e are the semimajor axis, focal parameter and eccentricity; $V_{cr}(r)$ is the angular velocity at the distance r from the center of gravitation; C is the constant of area; μ is the gravitational constant.

1. Formulation of the problems. It frequently is necessary to solve the following problems in the planning of orbits and impulse orbital transfers.

Problem 1. At point 1, three of the following four values are stipulated

$$r_1, V_1, \theta_1, \vartheta_1 \tag{1.1}$$

It is necessary to find the fourth value and the values

$$p_1, e_1, a_1 \tag{1.2}$$

and also the parameters (1.1) at point 2, which is given by one of the values (1.1) or by the angle**

$$\alpha = (\hat{r_1, r_2}) \quad (0 < \alpha < 2\pi). \tag{1.3}$$

Problem 2. At a stipulated point the spacecraft is imparted the velocity

*Numbers in the margin indicate pagination in the original foreign text.

**Henceforth the annotation $\beta = (\hat{a,b})$ means that when $\beta > 0$ the vector a must be turned in the direction of movement of the spacecraft (counter-clockwise in our figures) by the angle β so that it will coincide in direction with the vector b.

impulse ΔV lying in the orbital plane. It is necessary to determine the parameters (1.1) at this point after the impulse.

Problem 3. At the points 1 and 2 some of the values (1.1) and (1.3) are known, defining in finite values the orbit passing through these points. At these points it is necessary to find the values (1.1).

In the solution of these and other problems, a considerable saving of work results from use of geometrical methods ensuring an accuracy to the third to fourth place and in some cases indicating a simple approach to an analytical solution. In [1-3], the geometrical solutions of some problems were based on a special velocity hodograph. For problems 1-3 we will use the classical Hamilton hodograph (and in this sense we follow [4], where several variants of problem 1 were solved, giving simpler solutions).

The following theory [5] serves as a basis.

The velocity of a spacecraft moving in a Newtonian field of force can be broken down into two components, the moduli of which are invariable with time: V_A, perpendicular to the line of aspides, and V_R, perpendicular to the radius vector. We have

$$V_A = \frac{\mu e}{C}, \quad V_R = \frac{\mu}{C}, \quad (\hat{\mathbf{r}}, \hat{\mathbf{V}}_R) = (\mathbf{r}, \hat{\mathbf{V}}_A) = \frac{\pi}{2}. \tag{1.4}$$

This means that the velocity hodograph is a circle with the radius V_R, /1104 whose center is displaced relative to the origin of the hodograph by the vector V_A (Figure 1).

Then we assume that for a given Newtonian field we have the graph $V_{cr}(r) = \sqrt{\mu/r}$, which makes it possible to find r if $V_{cr}(r)$ is known, and vice versa. This puts the computations fully into geometrical form.

If the triangle of velocities V, V_A, V_R is known at some point of the orbit, within it we can geometrically determine any of the values (1.1) and (1.2), as follows from the equations:

$$\theta = (\hat{\mathbf{V}}, \hat{\mathbf{V}}_R), \quad \theta = (\hat{\mathbf{V}}_A, \hat{\mathbf{V}}_R), \quad e = \frac{V_A}{V_R}, \quad [V_{cr}(a)]^2 = V_R^2 - V_A^2, \quad V_{cr}(p) = V_R. \tag{1.5}$$

In addition, $V_{cr}(r)$ is the altitude dropped to the hypotenuse in ΔABD (Figure 2), in which $BC = V_R$ and $AC = V\cos\theta$ are the projections of the legs onto the hypotenuse, since

$$[V_{cr}(r)]^2 = V_R V\cos\theta. \tag{1.6}$$

The final objective in solution of problems 1-3 will be the construction of triangles of velocities at investigated points. We will give analytical solutions following from the geometry, when they are simpler than the solutions obtained directly from the classical formulas.

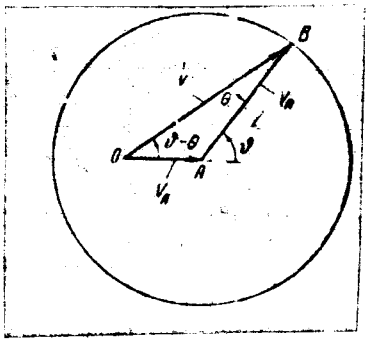

Figure 1.

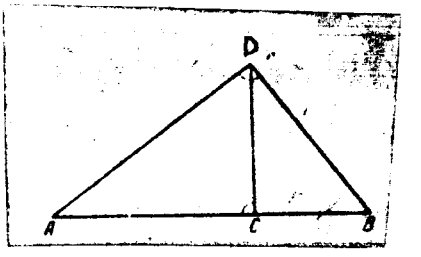

Figure 2.

2. Solution of problems 1 and 2. a) The values r, V, θ are stipulated. In ΔABD (Figure 2) we know AC = V cosθ and CD = $V_{cr}(r)$. This means V_R = BC, and the triangle of velocities V, V_A, V_R is constructed using two sides and the angle between them. Applying the theorem of sines in the triangle of velocities and expression (1.6) we have

$$\tan \vartheta = \frac{\sin \theta}{\cos \theta - \mu/rV^2 \cos \theta}. \qquad (2.1)$$

b) The values r, V, ϑ are stipulated. We determine BC = V_R cosθ, constructing ΔABD (Figure 2), in which AC = V, CD = $V_{cr}(r)$. Then on MN (Figure 3) we lay off AB = V and draw the rays AK and AL in such a way that $(\widehat{AB,AK}) = \pi/2$, $(\widehat{AL,AK}) = \vartheta$. From the middle of AB (AD = DB) we construct a perpendicular

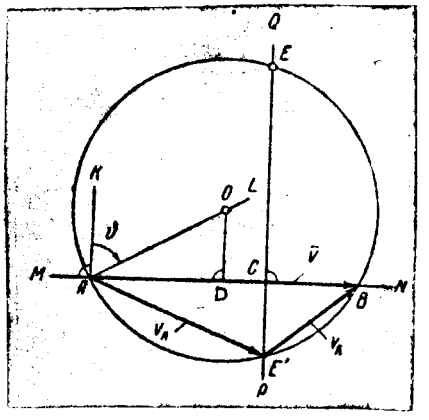

Figure 3.

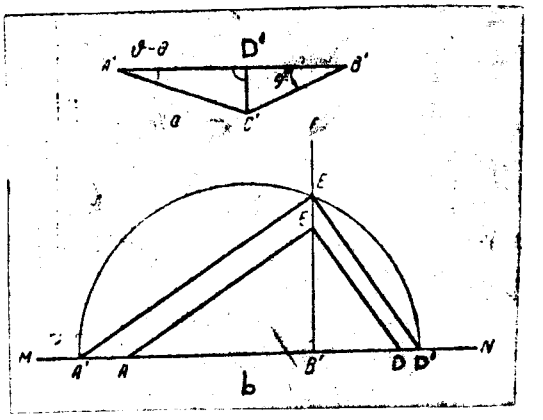

Figure 4.

to the intersection with the ray AL (or its extension through the point A) at/1105 the point O. We draw a circle of the radius OA. We lay off BC = $V_R \cos\theta$ from the point B in the direction of the point A and through the point C we draw PQ $\perp$ MN.

If $V_R \cos\theta < V$ (otherwise $V^2 > \mu/r$), the points of intersection of PQ with the circle lie on different sides of MN: E -- on the same side as the point K, and E' -- on the opposite side. It is easy to demonstrate that the triangle (V, V_A, V_R) is equal to $\Delta ABE'$, if $0 < \vartheta < \pi$, and equal to ΔABE, if $\pi < \vartheta < 2\pi$.

In the case $V_R \cos\theta > V$ there are two solutions when $\pi/2 < \vartheta < 3\pi/2$ and no solutions when $0 < \vartheta < \pi/2$, $3\pi/2 < \vartheta < 2\pi$. If it is taken into account that the straight line PQ must intersect with the circle, we obtain the following necessary and adequate conditions for the existence of solutions in general:

$$\frac{\pi}{2} < \theta < \frac{3\pi}{2} \text{ и } V_R \cos\theta - \frac{\mu}{rV} < \frac{V}{2}\left(1 + \frac{1}{|\sin\theta|}\right) \quad V^2 < \frac{2\mu}{r},$$
$$\theta \neq \pi \quad V^2 > \frac{2\mu}{r}. \tag{2.2}$$

and for two solutions, in particular:

$$\frac{2\mu}{r}\frac{|\sin\vartheta|}{1+|\sin\vartheta|} < V^2 < \frac{\mu}{r}, \quad \frac{\pi}{2} < \vartheta < \frac{3\pi}{2}. \tag{2.3}$$

c) The values r, θ, ϑ are stipulated. We construct $\Delta A'B'C'$ (Figure 4,a) similar to the triangle of velocities and we draw C'D' $\perp$ A'B' so that A'B' $\sim$ V, B'D' $\sim V_R \cos\theta$. On MN (Figure 4,b) we successively lay off the segments A'B' and B'D'; on A'D', as on a diameter, we construct a circle and draw B'F $\perp$ MN. On B'F we lay off B'E = $V_{cr}(r)$ and construct ΔADE, similar to $\Delta A'D'E'$. Since

$$\frac{A'B'}{V} = \frac{B'D'}{V_R \cos\theta} = \frac{B'E'}{V\ (r)}, \tag{2.4}$$

then AB' = V.

This solution leads to a simple formula (we use (1.6), (2.4) and the sine theorem in ΔABC (Figure 4,a)):

$$V = \sqrt{\frac{\mu}{r}\frac{\sin\vartheta}{\sin(\vartheta-\theta)\cos\theta}}. \tag{2.5}$$

The solution of variant (V, θ, ϑ) is obvious.

The second part of problem 1 is solved simply if it is taken into account

that

$$V_{A_2} = V_{A_1},\quad V_{R_2} = V_{R_1},\quad (V_{R_1}, V_{R_2}) = \alpha.$$

In the problem 2 the triangle of velocities (V^-, V_A^-, V_R^-) prior to the impulse is known. After the impulse

$$V^+ = V^- + \Delta V,\quad r^+ = r^-,\quad \theta^+ = (V^+, V_R^-),$$

if

$$-\pi/2 < (V^+, V_R^-) < \pi/2,\quad \theta^+ = \pi - (V^+, V_R^-),$$

if

$$\pi/2 < (V^+, V_R^-) < 3\pi/2$$

(in this case after the impulse there is a change in the direction of rotation about the center of gravitation).

3. Solution of problem 3. The combinations of parameters (1.1) and (1.3) at the points 1 and 2, unambiguously defining the orbit passing through these points, are reduced (with an accuracy to the numbering of the points) to the following:

1	$r_2V_2r_2\theta_2$	9	$r_1\theta_2V_2\theta_2$	15	$V_2\theta_1V_2\theta_2$	22	$\alpha r_1V_1\theta_2$
2	$r_1V_2r_2\theta_2$	9	$r_1\theta_1V_2\theta_2$	16	$V_1\theta_1V_2\theta_2$	23	$\alpha r_1\theta_1r_2$
3	$r_1V_1V_2\theta_2$	10	$r_1\theta_1\theta_2\theta_2$	17	$V_1\theta_1\theta_2\theta_2$	24	$\alpha r_1\theta_1V_2$
4	$r_1V_1V_2\theta_2$	11	$r_1\theta_1r_2\theta_2$	18	$V_1\theta_1V_2\theta_2$	25	$\alpha r_1\theta_2r_2$
5	$r_1V_1\theta_2\theta_2$	12	$r_1\theta_1V_2\theta_2$	19	$V_1\theta_1\theta_2\theta_2$	26	$\alpha V_1\theta_1r_2$
6	$r_1\theta_1r_2\theta_2$	13	$r_1\theta_1V_2\theta_2$	20	$\alpha r_1V_1r_2$	27	$\alpha V_2\theta_1V_2$
7	$r_1\theta_1r_2\theta_2$	14	$r_1\theta_1\theta_2\theta_2$	21	$\alpha r_1V_1V_2$	28	$\alpha V_1\theta_1\theta_2$

Below we cite the geometrical solutions of variants 5, 14, 19, 23, 25, /1106
27, 28. The variants 10 and 17 easily are reduced to problem 1. In cases 1-4, 6, 7, 11, 18 the solutions follow from formulas derived by use of the integrals of areas and energies and the known relations:

$$r = \frac{p}{1 + e\cos\vartheta}, \quad \operatorname{tg}\theta = \frac{e\sin\vartheta}{1 + e\cos\vartheta},$$
$$V = \sqrt{\frac{\mu}{p}\left[(e\sin\vartheta)^2 + (1 + e\cos\vartheta)^2\right]} \tag{3.1}$$

For the remaining variants no simple geometrical solutions (without construction of special curves) could be found, although the corresponding formulas can be derived from (3.1), (3.8) and the energies integral.

a) Variant $5(r_1 V_1 \vartheta_2 \theta_2)$. Assume $V_1^2 > 2\mu/r_1$. We find the value $\sqrt{V_2^2 - 2V_{cr_2}^2} = \sqrt{V_1^2 - 2V_{cr_1}^2}$ as the leg of a triangle in which the hypotenuse is equal to V_1 and another leg is equal to the hypotenuse of an isosceles triangle with the leg V_{cr_1}. We construct $\Delta A'B'C'$ (Figure 4, a), similar to the triangle of velocities at point 2,

$$A'B' = xV_2, \; A'C' = xV_A, \; B'C' = xV_R$$

(x is a proportionality factor). If we then construct ΔABD (Figure 2), in which $AC = A'B' \cos\theta_2 = xV_2 \cos\theta_2$, $BC = B'C' = xV_R$, then $CD = xV_{cr_2}$.

Then (Figure 5) we construct $GH = x\sqrt{V_2^2 - 2V_{cr_2}^2}$ ($FH = FE = xV_{cr_2}$, $EG = xV_2$). We lay off $HG' = \sqrt{V_2^2 - 2V_{cr_2}^2}$ and draw $E'G' \parallel EG$, $E'F' \parallel EF$. Then we have $E'G' = V_2$, $E'F' = V_{cr_2}$.

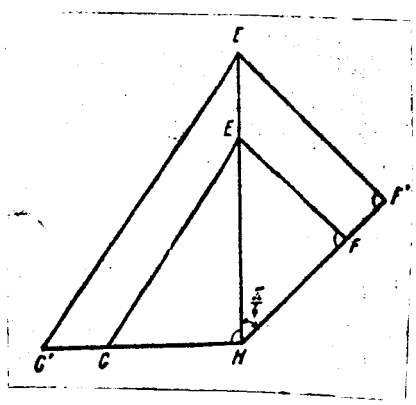

Figure 5.

The solution procedure is the same when $V_1^2 < 2\mu/r_1$. It only is necessary

to construct $\sqrt{2V_{cr_2}^2 - V_2^2}$, $x\sqrt{2V_{cr_2}^2 - V_2^2}$. The analytical solution follows from the equations:

$$V_2 = \sqrt{\frac{V_1^2 - 2\mu/r_1}{1 - 2(V_\mu/V_2)\cos\vartheta}} = \sqrt{\frac{V_1^2 - 2\mu/r_1}{1 - 2\sin(\vartheta_2 - \vartheta_2)\cos\vartheta_2 \sin\vartheta_2}} = \sqrt{\left(V_1^2 - \frac{2\mu}{r_1}\right)\frac{\sin\vartheta_2}{\sin(2\vartheta_2 - \vartheta_2)}}. \quad (3.2)$$

b) In variants 14 and 19 we construct $\Delta AOB \sim (V_2, V_{A_2}, V_{R_2})$ so that $OA \sim V_A$, $AB \sim V_R$, and $OB \sim V_2$ (Figure 6). Then we construct ΔACO, in which $OC = AC$

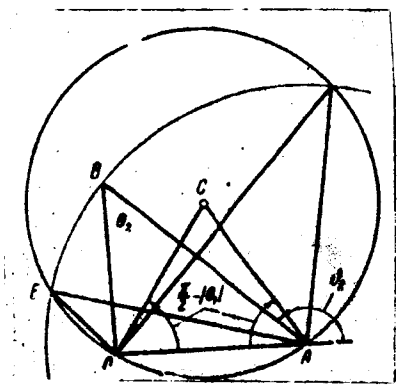

Figure 6.

and the angles at the base AO are equal to $\pi/2 - |\theta_1|$. In this case the point C should lie on the same side as the point B if the signs of θ_1 and θ_2 are identical and on the opposite side if the signs are different. We draw circles with the centers at the points A and C with the radii AB and AC respectively. D and E are the points of intersection of these circles. The triangles OAD and OAE are similar to the two variants of the triangle of velocities at point 1./1107 In actuality, $(DO\hat{;}DA) = \frac{1}{2}(CO\hat{;}CA) = \theta_1$ and $AB = AD = AE$. Then see problem 1.

The formula for ϑ_1 is derived from

$$\frac{\sin(\vartheta_1 - \theta_1)}{\sin\theta_1} = \frac{\sin(\vartheta_2 - \theta_2)}{\sin\theta_2}\left(= \frac{V_R}{V_A}\right). \quad (3.3)$$

c) Variant 23 ($\alpha r_1 \theta_1 r_2$).

Point 2 should lie on the same side of the straight line passing through the velocity vector at point 1 as the center of gravitation. It therefore follows that

$$r_2 < r_1 \frac{\cos\theta_1}{\cos(\alpha - \theta_1)} \quad \text{if} \quad 0 < \alpha < \frac{\pi}{2} - \theta_1 \quad \text{or} \quad \frac{3\pi}{2} - \theta_1 < \alpha < 2\pi \tag{3.4}$$

For the time being assume $0 < \alpha < \pi$. We intersect MN (Figure 7) by the

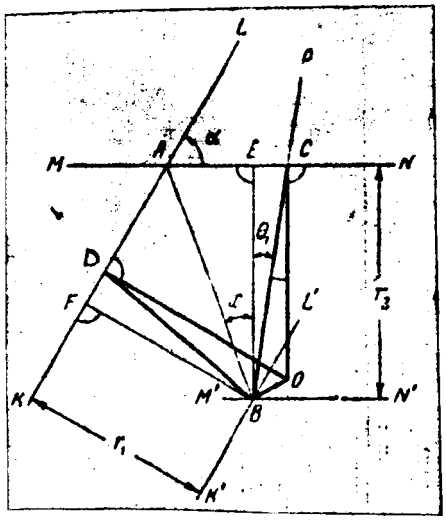

Figure 7.

straight line KL so that $(\hat{AN,AL}) = \alpha$. Then we draw $K'L' \parallel KL$ and $M'N' \parallel MN$ so that the distances between the parallel straight lines will be equal to r_1 and r_2 respectively (at some scale) and the point B of their intersection lies within the angle $(\hat{AK,AN}) = \pi - \alpha$. We draw $BE \perp MN$ and BP so that $(\hat{BP,BE}) = \theta_1$. The point C of intersection of BP and MN lies on the same side of A as N. In actuality,

$$AC = AB \cdot BC = BE(\operatorname{tg} x + \operatorname{tg}(\alpha - x)) \tag{3.5}$$

However, from the equations

$$BE = AB \cos x, \; r_2 BF = r_2 BE \cdot BP \cdot \cos\angle FBA = AB \cos(\alpha - x) \tag{3.6}$$

it follows that

$$\operatorname{tg} x = \left(\frac{r_2}{r_1} - \cos\alpha\right) \frac{1}{\sin\alpha} \tag{3.7}$$

and taking into account (3.4) AC $\geqslant$ 0.

Then on AK we lay off AD = AC and draw

$$DO \perp KL \text{ и } CO \perp MN,$$
$$\triangle BOC \sim (V_1, V_{A_1}, V_{R_1}),$$
$$\triangle BOD \sim (V_2, V_{A_2}, V_{R_2}).$$

We will derive a formula for V_1. From Figure 7, with (1.6), (3.5) - (3.7) taken into account, we have

$$\frac{V_1}{V_R} = \frac{r_1 V_1^2 \cos\theta_1}{\mu} = \frac{BC}{CO} = \frac{BC}{AC \operatorname{ctg}\frac{\alpha}{2}} = \frac{BB}{BE\cos\theta_1\left[\operatorname{tg}\theta_1 + \left(\frac{r_1}{r_2} - \cos\alpha\right)\frac{1}{\sin\alpha}\right]\operatorname{ctg}\frac{\alpha}{2}},$$

$$V_1 = \frac{\sin\frac{\alpha}{2}}{\cos\theta_1}\sqrt{\frac{2\mu}{r_1}\frac{r_2\cos\theta_1}{r_1\cos\theta_1 - r_2\cos(\alpha+\theta_1)}}. \qquad (3.8)$$

The case $\pi < \alpha < 2\pi$ reduces to the preceding case since the value V_1 remains as before if the angle θ_1 is replaced by the angle $-\theta_1$ and the angle α by the angle $2\pi - \alpha$. The formula (3.8) is suitable for any α. If $\alpha = \pi$, then

$$V_1\cos\theta_1 = \sqrt{\frac{2\mu}{r_1}\frac{r_2}{r_1+r_2}} = \sqrt{2}\,V_{cr_2}\frac{V_{cr}(r_1+r_2)}{V_{cr_2}}.$$

Hence follows a geometrical solution in this case as well (the preceding solution is unsuitable here).

The described problem was solved geometrically in [6] by other methods and only for elliptical orbits. The solution given here is considerably simpler and suitable for any orbits.

d) Variants 25 ($\alpha r_1\theta_1\theta_2$) and 28 ($\alpha V_1\theta_1\theta_2$). We construct directly tri- /1108
angles which are similar to the triangles of velocities at points 1 and 2. The formula for ϑ_1 is obtained using the sine theorem

$$\operatorname{tg}(\vartheta_1 - \theta_1) = \frac{\sin\theta_1\sin(\alpha+\theta_1-\theta_2)}{\sin\theta_2 - \sin\theta_1\cos(\alpha+\theta_1-\theta_2)}.$$

e) Variant 27 ($\alpha V_1 \theta_1 V_2$). On MN (Figure 8) we lay off $OB = V_1$. Then we draw BP and BQ in such a way that $(\widehat{BM;BP}) = \theta_1$, $(\widehat{BQ;BP}) = (\pi - \alpha)/2$. The circle with the radius V_2, drawn from the center O, intersects BQ at the points D_1 and D_2. We draw $D_1A_1 \parallel D_2A_2$ so that $(\widehat{D_1A_1;D_1B}) = (\pi - \alpha)/2$. The problem has two solutions (they merge into one if $OD_1 = OB \cdot \sin \angle OBQ$):

1) $\triangle OA_1B = (\mathbf{V}_1, \mathbf{V}_{A_1}, \mathbf{V}_{R_1})$, $\triangle OA_1D_1 = (\mathbf{V}_2, \mathbf{V}_{A_1}, \mathbf{V}_{R_1})$,
2) $\triangle OA_2B = (\mathbf{V}_1, \mathbf{V}_{A_2}, \mathbf{V}_{R_2})$, $\triangle OA_2D_2 = (\mathbf{V}_2, \mathbf{V}_{A_2}, \mathbf{V}_{R_2})$.

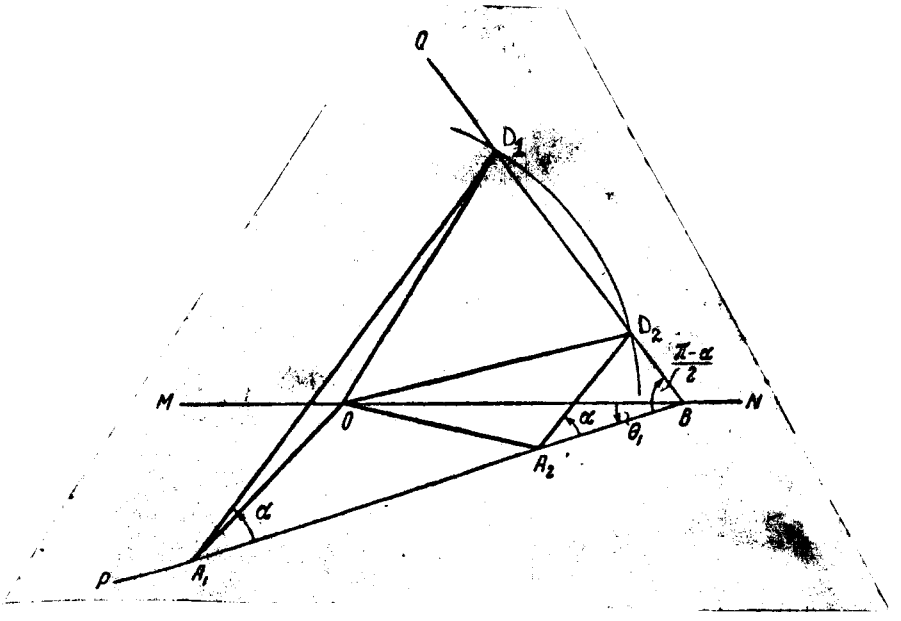

Figure 8.

For existence of the solutions it is necessary and sufficient that the circle intersects BQ, that is

$$V_2 \geqslant V_1 \left| \cos\left(\frac{\alpha}{2} + \theta_1\right) \right|.$$

From ΔOBD_2, using the sine theorem, we have two values of the angle θ_2:

$$\cos\left(\frac{\alpha}{2} - \theta_2\right) = \frac{V_1}{V_2} \cos\left(\frac{\alpha}{2} + \theta_1\right).$$

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